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Algorithms for the free-surface Green function

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Summary

Numerical methods are outlined for computing the velocity potential, and its derivatives, for linearized three-dimensional wave motions due to a unit source with harmonic time dependence beneath a free surface. Two distinct cases are considered where the fluid depth is either infinite, or of constant finite depth. Efficient algorithms are developed in both cases, to replace the numerical evaluation of the relevant integrals by multi-dimensional approximations in economized polynomials. This technique is substantially faster than conventional direct methods based on numerical integration.

1. Introduction

The source-potential, or Green function, is the fundamental element in the analysis of wave-induced motions and forces acting on floating or submerged vessels. In the case of most practical importance, a numerical model is based on distributions of sources, and optionally of higher-order singularities analogous to dipoles, located on the submerged portion of the body surface. This procedure, which can be justified by Green's theorem, requires the solution of an integral equation in the domain of the body surface, either for the source strength or for the velocity potential. In practice, the body surface is discretized in an appropriate manner, and the integral equation is replaced by a finite system of linear equations.

Two distinct numerical problems must be overcome to implement this approach successfully for a fully three-dimensional body geometry. First, to describe the body surface with a reasonable degree of fidelity, a large number of discrete "panel" elements must be utilized, typically between 100 and 1000. The corresponding linear system of equations is characterized by a square matrix of complex coefficients with the same dimension, which must be solved by a suitable application of linear algebra. Numerous subroutines are available to perform this task efficiently.

The second numerical problem, peculiar to the field of free-surface hydrodynamics, is the evaluation of the source potential and its derivatives. These are complicated mathematical functions, which must be evaluated successively for each combination of panels, i.e., between 10^4 and 10^6 times for each body geometry and frequency of wave encounter. Since a realistic description of the wave-body interactions dictates that this computation should be repeated for 10–100 frequencies, between 10^5 and 10^8 evaluations of the source potential are required to analyse a single vessel. This is regarded as the main difficulty in performing three-dimensional computations of hydrodynamic parameters such as the motions of a body in waves, or of the pressure forces exerted on the body in the same environment.

Mathematical expressions for the oscillatory source potential are well known. The standard reference for these functions is Wehausen and Laitone [1]; recent references to the same expressions are Sarpkaya and Isaacson [2], Susbielles and Bratu [3], and Mei [4]. The classical representation is in terms of a semi-infinite integral involving a Bessel function and a Cauchy singularity. Separate expressions exist for infinite and finite (constant) depth of the fluid, but their forms are similar and the infinite-depth limit can be recovered as a special case of the finite-depth integral representation. The principal drawback of these expressions is that they are extremely time-consuming to evaluate numerically.

In the case of infinite depth, a simpler analytic representation for the source potential exists as the sum of a finite integral, with a monotonic integrand involving elementary transcendental functions, and a wave-like term of closed form involving Bessel and Struve functions. This expression, which was suggested by Havelock [5], has been rederived or publicized by Kim [6], Hearn [7], Liapis and Dahle [8], Noblesse [9], and Newman [10]. In the case of finite depth an analogous alternative to the conventional integral representation is due to John [11] in the form of a discrete eigenfunction expansion.

Havelock's and John's expressions for the source potential are complementary, with the former valid only for infinite depth and the latter for finite depth. But despite the fundamental differences in their derivations and analytic forms, these two alternatives to the conventional integral representations have common attributes including (1) avoidance of the relatively complicated numerical analysis of the conventional integrals, and (2) an analytical decomposition of the source potential into an oscillatory wave-like component of relatively simple form, plus a local component which oscillates at most a finite number of times. However, (3) John's series and Havelock's integral are singular along the vertical axis which coincides with the source point, and these formulae are numerically inefficient within a finite radius of that axis.

This paper describes efficient numerical algorithms for computing the oscillatory source potential, and all relevant derivatives of this function, throughout the domain of physical importance. We start from the premise that numerical integration should be avoided in all cases. Analytical series expansions are used where these are computationally efficient, and in the remaining subdomains methods are described for deriving systematic multi-dimensional polynomial approximations to provide direct evaluations of the source potential.

The Bessel functions $J_n(x)$ and $Y_n(x)$ provide familiar examples to illustrate our philosophy. One might define these functions initially by their integral representations, which involve oscillatory integrands and either a finite or infinite range of integration, respectively. Numerical integration provides a possible algorithm for the evaluation of these functions, but with a relatively large computational burden due to the time required for many evaluations of the trigonometric functions in the integrands, and due to relatively slow convergence of the infinite integral for Y_n. Instead, standard algorithms for computing these special functions establish a partition point at some finite value of x, and utilize analytic properties of the functions to develop complementary algorithms on each side of the partition. For small values of x the ascending series expansions are computationally efficient, and can be refined by the technique of polynomial economization. For large values of x the asymptotic expansions are not sufficiently accurate by themselves, but these provide guidance for suitable forms of polynomial approximations, in descending powers of the argument x, for the modulus and phase of these functions. In this manner an efficient pair of algorithms is obtained for all values of x, involving relatively simple polynomial expressions and a few evaluations of the elementary transcendental functions. (Examples of the results are contained in Abramowitz and Stegun [12]; more refined approximations and an outline of the above procedure are given by Newman [13].)

Similar techniques can be applied to a function of two or more variables, such as the source potential. As in the one-parameter case, the first step in developing an efficient computational algorithm is to understand the analytical behavior of the relevant function, and to separate singular or rapidly oscillating elements from the part to be approximated. In the present context this includes the study of analytic expansions, such as John's finite-depth infinite series, which can be used directly in domains where they are most effective, or alternatively to indicate the forms for polynomial approximations in the subdomains where the expansions themselves are not useful. Multi-dimensional Chebyshev expansions provide an efficient basis for developing the polynomial approximations to any desired degree of accuracy.

In infinite depth the portion of the source potential excluding the elementary singularity $1/r$, where r is the distance between the source and field points, can be expressed as a nondimensional function of two independent parameters. These two nondimensional coordinates are the radial and vertical separation between the field point and the image source point above the free surface, normalized by the wavenumber. Since these two coordinates may take on all positive values, one quadrant of a two-dimensional plane must be considered. Three different series expansions are numerically efficient around the boundaries of this quadrant, and can be complemented by an asymptotic expansion for large radial distance from the origin. This leaves a central domain where polynomial approximations in two variables are required. The asymptotic expansion suggests an appropriate form to assume in the central domain, with a residual correction which is amenable to polynomial representation. Once the coefficients of this polynomial are derived, the source potential and its derivatives may be evaluated for all points in the quadrant.

Complementary developments are considered in Section 3 for the finite-depth case. Here John's expansion is effective provided the horizontal component R exceeds half the depth h. Returning to the integral expression in the converse domain, the most obvious difficulty in developing polynomial approximations is that the source potential is a function of four nondimensional parameters. The degree of this difficulty is reduced by expressing the potential as the sum of two similar functions, each depending on only three parameters. Finally, prior to deriving polynomial approximations in three variables, an appropriate combination of infinite-depth source potentials is subtracted from the finite-depth integral to accelerate the convergence of the integral itself, and to leave a nonsingular remainder. With the three-dimensional polynomials and appropriate combinations of infinite-depth potentials used for $R/h < 1/2$, and the infinite-series representation used otherwise, the finite-depth source potential and its derivatives may be evaluated for all values of the relevant coordinates.

The ultimate accuracy required is an important factor in any numerical computation, and especially in the development of polynomial approximations. An arbitrary degree of precision can be attained, in principle, but it is obvious that relaxing the tolerance of the computations will lead to simpler approximations, smaller storage requirements and faster running times. Here, guided by the widespread use of computing machines with single-precision accuracy of 6–7 decimals, an absolute or relative accuracy of $6D$ has been selected for all approximations of the source potential and its first derivatives, exclusive of the elementary singularity $1/r$. No attempt is made to preserve relative accuracy when a computed quantity is of small magnitude, since its contribution to the integrals over the body surface can be assumed to be correspondingly small.

With this tolerance in the coefficient matrix, a high degree of precision can be maintained in the ultimate solution of the linear system of equations and the dominant numerical errors may be associated with the discretization of the body surface. The latter tolerance can be controlled, by systematically increasing the number of panels, and the accuracy of the final hydrodynamic parameters can then be judged with some confidence. In special circumstances other choices of precision may be appropriate for the evaluation of the source potential, and one feature of systematic polynomial approximations is that these can be derived to any desired degree of accuracy.

The following sections describe sets of algorithms for the source potential which have been found to be efficient with the above accuracy. Specific computing times are cited in Section 4 to confirm the value of this approach.

In all cases the derivatives follow directly by analytic differentiation of the corresponding algorithm. In the infinite-depth case only the first horizontal derivative must be computed, since the vertical derivative can be expressed as a linear combination of the potential and the inverse-square-root, and all higher-order derivatives can be derived by algebraic combinations of the source potential and its horizontal derivative.

Throughout this work the complex time-dependent factor $\exp(i\omega t)$ is assumed, and our primary concern is to evaluate the real part of the corresponding complex factor $G(x, y, z; \xi, \eta, \zeta)$. Here (x, y, z) is a Cartesian coordinate system defining the field point, (ξ, η, ζ) is the location of the source in terms of the same coordinates, $z = 0$ is the plane of the undisturbed free surface and the fluid domain corresponds to negative values of z. The vertical position of the source is assumed to be negative or zero. The fluid depth is assumed to be either infinite, or of constant depth h. The frequency and gravitational acceleration g define the parameter $K = \omega^2/g$, which corresponds to the wavenumber in the infinite-depth case and will be used more generally to nondimensionalize all length scales. Anticipating axisymmetry about the vertical axis coincident with the source, the only nontrivial horizontal coordinate is the radius R, defined by the magnitude of the horizontal vector with components $(x - \xi, y - \eta)$.

2. The infinite-depth case

In its conventional form (cf. Wehausen and Laitone [1], equation 13.17) the source potential is defined by the expression

$$G = \left[R^2 + (z - \zeta)^2\right]^{-1/2} + \int_0^\infty \frac{k + K}{k - K} e^{k(z+\zeta)} J_0(kR) \mathrm{d}k. \tag{1}$$

Here J_0 denotes the Bessel function of the first kind, order zero, and the contour of integration passes above the pole to satisfy the radiation condition of outgoing waves at infinity. In terms of the two nondimensional coordinates

$$X = KR, \quad Y = K|z + \zeta|$$

(1) may be written in the form

$$G = \left[R^2 + (z - \zeta)^2\right]^{-1/2} + KF(X, Y) - 2\pi i K e^{-Y} J_0(X). \tag{2}$$

The principal task here is to evaluate the nondimensional real function $F(X, Y)$ for all relevant values of the parameters (X, Y) of possible physical interest. Rewriting this function from (1), and also in the alternative form of Havelock's finite integral (cf. [10]), two equivalent definitions for F are given by

$$F(X, Y)=\int_0^\infty \frac{k+1}{k-1} e^{-kY} J_0(kX) \mathrm{d}k, \tag{3a}$$

$$= (X^2+Y^2)^{-1/2} - \pi e^{-Y}(\mathbf{H}_0(X)+Y_0(X)) - 2\int_0^Y e^{t-Y}(X^2+t^2)^{-1/2} \mathrm{d}t. \tag{3b}$$

In the latter expression Y_0 is the Bessel function of the second kind, and $\mathbf{H}_0$ denotes the Struve function as defined by Abramowitz and Stegun [12] (Chapter 12). These functions of one variable, and the corresponding Bessel function J_0 which is required for the imaginary part of (2), can all be approximated by standard methods. Algorithms for these particular special functions which are efficient in the present context are developed by Newman [13]. In (3a) the integral is defined as the Cauchy principal value.

Equation (3a) is the most convenient starting point for small values of X, since the Bessel function in this integral can be expanded in even powers of kX and integrated term-by-term. The result is a double infinite series with positive powers of X and negative powers of Y, first derived by Hess and Wilcox [14] and subsequently by Noblesse [9]:

$$F(X, Y) = (X^2+Y^2)^{-1/2} + 2\sum_{n=0}^{\infty} \frac{(-X^2/4)^n}{(n!)^2} \left(\sum_{m=1}^{2n} \frac{(m-1)!}{Y^m} - e^{-Y} Ei(Y) \right). \tag{4}$$

The exponential integral $Ei(Y)$ ([12], Chapter 5) can be approximated in a straightforward manner, since it is a function of only one variable. Useful continued-fraction and rational-fraction approximations for this function are given by Cody and Thacher [15]. The remaining summation in (4) is straightforward, in the domain of convergence $X < Y$. For $6D$ accuracy this series can be truncated with $n \leqslant 9$, throughout the domain $X/Y < 0.5$.

An alternative series expansion can be derived from (3b), by expanding the last exponential function in powers of t and integrating term-by-term. The result, derived by Newman [16], differs from (4) insofar as it involves positive powers of both coordinates (X, Y) and is uniformly convergent throughout the full domain of interest:

$$F(X, Y) = (X^2+Y^2)^{-1/2} - 2 e^{-Y}\Big(J_0(X) \log\big(Y/X + (1+Y^2/X^2)^{1/2}\big) + \frac{\pi}{2} Y_0(X) + \frac{\pi}{2X} \mathbf{H}_0(X)(X^2+Y^2)^{1/2} + (X^2+Y^2)^{1/2} \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} C_{mn} X^{2m} Y^n \Big). \tag{5}$$

Here the coefficients C_{mn} are defined by

$$C_{0n} = [(n+1)(n+1)!]^{-1},$$

$$C_{mn} = -\left(\frac{n+2}{n+1}\right)C_{m-1,n+2}$$

The double series in (5) can be truncated and economized in both variables; 6D accuracy is achieved throughout the domain $0 < X < 3.7$, $0 < Y < 2$ with a total of 33 terms.

We next consider the case where X/Y is large. Here an asymptotic expansion can be derived in a straightforward manner, by expanding the inverse square-root in the integrand of (3b) in even powers of the ratio t/X, and integrating term-by-term. Thus

$$\frac{1}{X}\int_0^Y e^{t-Y}(1+t^2/X^2)^{-1/2}\mathrm{d}t$$

$$\simeq \frac{1}{X}\sum_{n=0}^{N}(-)^n\frac{X^{-2n}}{n!}\left(\tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tfrac{5}{2}\cdot\,\ldots\,\cdot(n-1/2)\right)I_{2n}(Y) \tag{6}$$

where

$$I_0 = 1 - e^{-Y}, \quad I_{2n} = \int_0^Y e^{t-Y}t^{2n}\mathrm{d}t.$$

Partial integration of the last integrals yields the recursion formula

$$I_{2n} = Y^{2n} - 2nY^{2n-1} + 2n(2n-1)I_{2n-2}.$$

Truncation of (6) with $N = 3$ gives $6D$ accuracy in the domain $X > 3.7$ provided $X/Y > 4$. The same accuracy can be achieved throughout the domain $2 < X/Y < 4$ if the descending series (6) is transformed to a continued fraction.

Finally we consider the asymptotic expansion of (3b) when both X and Y are large. Partial integration may be employed to derive the relation

$$\int_0^Y e^{t-Y}(X^2+t^2)^{-1/2}\mathrm{d}t$$

$$= \sum_{m=0}^{M}(-)^m\left\{\frac{\mathrm{d}^m}{\mathrm{d}Y^m}(X^2+Y^2)^{-1/2} - e^{-Y}\left[\frac{\mathrm{d}^m}{\mathrm{d}t^m}(X^2+t^2)^{-1/2}\right]_{t=0}\right\}$$

$$+(-)^M\int_0^Y e^{t-Y}\frac{\mathrm{d}^M}{\mathrm{d}t^M}(X^2+t^2)^{-1/2}\mathrm{d}t. \tag{7}$$

In the domain $Y > 20$ and $X > 8$ the summand proportional to $\exp(-Y)$ is negligible, and the last integral in (7) also is negligible if $M = 4$. The remaining terms in (7) can be expressed more directly in terms of spherical harmonics, and the desired result follows in the form

$$\int_0^Y e^{t-y}(X^2+t^2)^{-1/2}\mathrm{d}t \cong \sum_{n=0}^{4} n!P_n(\sin\theta)r^{-(n+1)} \tag{8}$$

where P_n is the Legendre polynomial, and (r, θ) denote polar coordinates such that $X = r\cos\theta$, $Y = r\sin\theta$. This asymptotic expansion gives $6D$ accuracy throughout the domain $X > 8$, $Y > 20$.

At this stage relatively simple expansions have been developed for all but a central domain of the X, Y quadrant. The remaining task is to seek numerical approximations more directly, for the domain $2 < Y < 20$ with X/Y intermediate between the regions of applicability of (4) and (6). After some numerical experimentation, guided by the intermediate form of the asymptotic expansion (7), the following form has been found to give a slowly-varying residual function $R(X, Y)$, which can be approximated effectively by two-dimensional polynomials:

$$\int_0^Y e^{t-Y}(X^2+t^2)^{-1/2}\,dt = (X^2+Y^2)^{-1/2} - e^{-Y}/X + Y(X^2+Y^2)^{-3/2}R(X, Y). \tag{9}$$

The numerical procedure used for this purpose can be described briefly as follows. The function $R(X, Y)$ is evaluated for arbitrary points in the relevant domain by integrating the finite integral in (9) with double-precision accuracy in the manner outlined in [10]. Double Chebyshev polynomial expansions are then generated for this function, and truncated to the desired accuracy by neglecting all coefficients larger than a corresponding tolerance. Conversion of the Chebyshev expansions to finite double series in ordinary powers of X and Y facilitates the subsequent routine use of the algorithm. It has been found that subdividing the central domain at $Y = 4$ and again at $Y = 8$ enables $R(X, Y)$ to be approximated in each of the three subdomains with a maximum of 37 nonzero polynomial coefficients.

Summarizing the algorithms described above, for the case of infinite depth, the source potential and its derivatives are expressed in separate subdomains of the X, Y quadrant, with an appropriate form of approximation in each subdomain. Most of these approximations are in the form of double polynomials in powers of X and Y, with coefficients which are determined separately in each subdomain. Typically there are 30–40 terms in these polynomials, or a corresponding number of floating point operations in the evaluations based on (4) or (6). (In practice, the ultimate choice of boundaries between these separate subdomains has been based on the objective of similar computing time in each regime.)

3. The finite-depth case

In a fluid of constant finite depth h, the source potential can be expressed in terms of a contour integral analogous to (3a),

$$G = \left[R^2 + (z-\zeta)^2\right]^{-1/2} + \left[R^2 + (2h+z+\zeta)^2\right]^{-1/2} + 2\int_0^\infty \frac{(k+K)\cosh k(z+h)\cosh k(\zeta+h)}{k\sinh kh - K\cosh kh} e^{-kh} J_0(kR)\,dk \tag{10a}$$

or, alternatively, John's [11] infinite-series expansion

$$G = 2\pi \frac{K^2 - k_0^2}{k_0^2 h - K^2 h + K} \cosh k_0(z+h) \cosh k_0(\zeta + h)[Y_0(k_0 R) + \mathrm{i} J_0(k_0 R)]$$

$$+ 4 \sum_{n=1}^{\infty} \frac{k_n^2 + K^2}{k_n^2 H + K^2 H - K} \cos k_n(z+h) \cos k_n(\zeta + h) K_0(k_n R). \qquad (10b)$$

Equations (10a, b) are given by Wehausen and Laitone ([1], equations 13.18, 13.19, respectively, with a typographical correction in the latter). In (10b) the wavenumber k_0 is defined as the positive real root of the transcendental equation $k \tanh(kh) = K$, and k_n denotes the set of corresponding imaginary roots, defined more explicitly as the positive real roots of the equation $k \tan(kh) = -K$. K_0 is the modified Bessel function of the second kind.

In estimating the numerical utility of the series (10b) it is important to note (with an obvious ordering of the roots) that

$$\pi(n - \tfrac{1}{2}) \leqslant k_n h \leqslant \pi n$$

and, for large n,

$$K_0(k_n R) = O(\exp(-\pi n R/h)).$$

Thus the rate of convergence of (10b) depends primarily on the ratio R/h, and the number of terms required for a given accuracy is proportional to h/R. The series (10b) is practically useless for small values of R/h, since each summand contains a logarithmic singularity when $R/h = 0$. Numerical experience confirms these estimates, and $[6h/R]$ is found to be an appropriate number of terms in the series to achieve $6D$ accuracy in the domain $R/h > 1/2$. (Some improvement can be affected in the convergence for small R/h by subtracting a simpler series with the same asymptotic form for large n, cf. equation 8.526 of Gradshteyn and Ryzhik [17]. This identity is useful in relating the limiting form of (10) for $Kh \to 0$ to an image system of elementary sources, but for small finite values of Kh the logarithmic behavior of each term is significant and cannot be removed in this manner.)

A complementary analysis for smaller values of R/h may be based on the integral expression (10a). For this purpose nondimensional parameters are introduced as follows:

$$X = KR, \quad Y = K|z|, \quad Z = K|\zeta|, \quad H = Kh.$$

Utilizing the product formula for two hyperbolic cosines, the real part of (10a) is expressed following John [11] in the form

$$\mathrm{Re}(G) = KL(X, |Y - Z|, H) + KL(X, 2H - Y - Z, H) \qquad (11)$$

where the auxiliary function L is defined by

$$L(X, V, H) = (X^2 + V^2)^{-1/2} + \int_0^{\infty} \frac{(k+1)\cosh(kV)}{k \sinh kH - \cosh kH} \mathrm{e}^{-kH} J_0(kX) \mathrm{d}k. \qquad (12)$$

Since Y and Z are restricted to the fluid domain $(0, H)$, the vertical coordinate V in (12) lies in the interval $(0, 2H)$. Note that one function of four independent parameters (X, Y, Z, H) has been replaced in (11–12) by the sum of two functions, each depending on only three parameters (X, V, H). Anticipating the ultimate approximation of these functions in terms of multi-dimensional polynomials, this represents a major simplification.

The rate of convergence of the integral in (12) can be accelerated by adding and subtracting an appropriate function which is asymptotically equivalent to the integrand for large k. A judicious choice for this function leads to the result

$$L(X, V, H)$$
$$= (X^2 + V^2)^{-1/2} + F(X, 2H - V) + F(X, 2H + V)$$
$$+ \int_0^\infty \left\{ \frac{1}{k \sinh kH - \cosh kH} - \frac{2\,\mathrm{e}^{-kH}}{k-1} \right\} (k+1) \cosh(kV)\, \mathrm{e}^{-kH} J_0(kX) \mathrm{d}k. \tag{13}$$

As k tends to infinity the integrand in (13) is of order $\exp(-2kH)$ or smaller. The two additional functions F in this decomposition are the integrals of the two portions of the function subtracted from the integrand which correspond to the expansion of the hyperbolic cosine $\cosh(kV)$ in a pair of exponential functions. In view of (3a), these extra integrals are identical to the infinite-depth source potential with indicated values of the vertical coordinate. (A more obvious decomposition could be made with only the dominant exponential term retained from the hyperbolic cosine, and with only the first of the two functions F added, but the resulting difference integral is not even in V, and thus is more difficult to approximate in polynomials.)

It is pertinent to note that the integral in (13) is regular near $X = 0$ for all relevant values of V. Thus the singular behavior of the finite-depth source potential for small values of the physical coordinates is identical to the infinite-depth potential, as displayed in (5). This observation extends the asymptotic approximation of (13) given by John [11], and corresponds with the physical interpretation that if the source and field points are sufficiently close to each other, and to the free surface, the singular part of the potential is not affected by the finite depth. More significantly, in the present context, the regular behavior of the integral which remains in (13) implies that it can be expanded as a polynomial in even powers of the coordinates X and V, multiplied by coefficients which depend on H. Finally, in a preliminary assessment of (13), the exponential bound on the integrand for large kH implies that the latter coefficients will tend to zero algebraically for large H. Except possibly for $H = 0$, these coefficients should be well behaved and expressible as polynomials in the depth H or its inverse. Thus the integral which remains in (13) is an effective form to approximate by three-parameter polynomials in (X, V, H).

An ad hoc approach has been followed to approximate the integral in (13). In the first step the Bessel function $J_0(kX)$ is expanded in even powers of kX. Preliminary computations reveal that for the domain $X/H < 0.5$ this expansion can be truncated with six terms, corresponding to the even powers 0 through 10. A double-precision routine is used to compute this family of six integrals, as functions of (V, H), with contour integration utilized in the complex k-plane to avoid the loss in accuracy inherent with a conventional numerical integration scheme in the vicinity of the two poles. The six integrals are approximated by double Chebyshev expansions in a manner analogous to that described for the function $R(X, Y)$ in Section 2. Finally, triple Chebyshev expansions are constructed from the monomials in X, economized, and converted to triple polynomials of the desired form in X, V, and H. The highest powers of X and V which are required after economization are 8 and 14, respectively. With partitions at $H = 2$ and $H = 4$, approximately 300 coefficients are needed in each subdomain. In the usual application where many evaluations are required with different values of X, V, for the same H, the polynomials in H can be evaluted and stored as a total of 33 nonzero coefficients for the remaining two-parameter polynomials in X and V.

Finally, the singular behavior of the integral in (13) should be noted in the limit $H \to 0$. After an asymptotic analysis it is possible to show that

$$\int_0^\infty k^{2n}(k+1)\cosh(kV)\,\mathrm{e}^{-kH}\left\{\frac{1}{k\sinh kH - \cosh kH} - \frac{2\,\mathrm{e}^{-kH}}{k-1}\right\}\mathrm{d}k$$

$$= \frac{-2\left(k_0^2 - H\right)k_0^{2n}}{k_0^2 H - H^2 + H}\cosh(k_0 V)\log(K_0 H) - 4\,\mathrm{e}^{-2H}\cosh H \log H + \ldots. \tag{14}$$

Here the neglected terms $(\ldots)$ are regular in H. Sufficiently well-behaved Chebyshev expansions result by subtracting the right side of (14) only for $n = 0, 1, 2$, and approximating the hyperbolic functions by truncated Taylor series. In this manner the ultimate polynomial form for the integral in (13) is retained, in the subdomain $0 < H < 2$, with a simple correction affected for six of the 33 elements in the (X, V)-polynomial coefficient matrix.

4. Conclusions

A set of algorithms has been described for evaluating the oscillatory free-surface source potential and its derivatives, in infinite or finite depth. The implementation of these algorithms is a nontrivial computational task, in view of the various separate subdomains which must be included, and of the substantial number of polynomial coefficients which are required ultimately. Thus it is appropriate to report briefly on the practical results which follow from this approach so that it can be appreciated as an alternative to the conventional numerical integration of (3) or (10a).

For the infinite-depth case the algorithms described in Section 2 have been implemented in a Fortran subroutine "NEWGREEN". This subroutine contains 332 lines of source code, exclusive of comments, and is self-contained except for the intrinsic functions (square-root, sine, cosine, log, and exponential). The two-dimensional polynomial coefficients, and coefficients of the various approximations required for the exponential integral, Bessel, and Struve functions require a total of 261 constants in DATA statements. For one evaluation of the (complex) source potential and its derivatives, the maximum runtime of this subroutine is approximately 400 microseconds on an IBM 370/168 and 800 microseconds on a VAX 11/782. (By comparison, the execution time on the VAX for one evalution of the Bessel function $J_0(x)$ using the library subroutine MMBSJ0 of the International Mathematical and Statistical Library (IMSL) is between 200 and 650 microseconds depending on the value of x.)

For the finite-depth case an analogous subroutine package "FINGREEN" has been developed from the algorithms in Section 3. This is a larger program, with 880 lines of source code, excluding comments, and 1301 coefficients in DATA statements. A maximum of 12 milliseconds is required for one call on the VAX, but the time for subsequent evaluations with the same value of $Kh = H$ is reduced to between 3 and 6 milliseconds. Thus, in the usual application to a boundary-integral program, the average runtime for the evaluation of the coefficient matrix in finite depth should be only about six times greater than for infinite depth. In all cases described both the potential and its first derivatives are evaluated simultaneously, with at least six decimals accuracy.

Similar algorithms based on the combination of series expansions and polynomial approximations may be applicable to other types of Green functions, including the case of a steady-state source moving with constant velocity beneath a free surface, and possibly also including Green functions of complicated mathematical form which arise in other branches of applied mechanics.

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